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An alternative approach to existence result of solutions for the Navier-Stokes equation through discrete Morse semiflows

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1 Discrete Morse semiflows

In this note we shall construct weak solutions to the Navier-Stokes equation by use of idea of the discrete Morse semiflows. First the concept of discrete Morse semiflows is explained.

Let I be a functional on some Banach space X. To find critical points we must solve the Euler-Lagrange equation

$$\delta I(u)=\left.\frac{d}{d\varepsilon}I(u+\varepsilon\varphi)\right|_{\varepsilon=0}=0 \tag{1.1}$$

for any $\varphi\in X$. We sometimes had better regard critical points as stationary points of flow defined by

$$u_t=-\frac{1}{2}\delta I(u), \tag{1.2}$$

which is called the Morse semiflow. In other words we solve (1.2) with some initial data u_0 and get solutions to (1.1) by passing to the limit as $t\to\infty$.

To solve this evolution equation we discretize (1.2) with respect to the time variable

$$\begin{cases} \dfrac{u_n^h-u_{n-1}^h}{h}=-\dfrac{1}{2}\delta I(u_n^h), \\ u_0^h \text{ is given.} \end{cases} \tag{1.3}$$

Now we assume $X\hookrightarrow L^2$. Since we can regard (1.3) as the Euler-Lagrange equation to

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the functional

$$J(u) = \frac{\|u - u_{n-1}^h\|_2}{h} + I(u),$$

where $\|\cdot\|$ is the L^2-norm, we can define u_n^h as the minimizer of this functional, *i.e.*,

$$u_n^h \ : \ J(u) \to \min.\ \text{in}\ X.$$

We call the sequence $\{u_n^h\}$ the *discrete Morse semiflow.* This idea can be found in a paper of Rektorys [7].

We conversely apply the idea of discrete Morse semiflows to evolution equations. Such trial had been done for the heat flow to harmonic maps by Bethuel-Coron-Ghidaglia-Soyeur [1], for a semilinear hyperbolic system by Tachikawa [9]. The author and Omata considered the asymptotics of discrete Morse semiflow for a functional with free boundary in [5, 6]. Here we shall apply the idea to the evolutional Navier-Stokes equation. The Navier-Stokes equation, however, is not an Euler-Lagrange equation to some functional, so we need some modification.

In the next section we shall regard the Navier-Stokes equation as an ordinary differential equation on some Banach space as usual manner, and define the concept of a *weak solution.* In § 3 we devote ourself to the explanation of our scheme. In § 4 we shall derive a priori estimates for an approximate solution, and construct a weak solution by vanishing the time increment of discretization. Furthermore we shall comment on our scheme in the last section.

2 Navier-Stokes equation

Let Ω be a domain in $\mathbf{R}^m$. We do not assume any smoothness of the boundary $\partial\Omega$. The initial-boundary value problem for the Navier-Stokes equation is described by

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla p = \Delta u + f & \text{in} \quad \Omega\times(0,\infty), \\ \text{div}\ u = 0 & \text{in} \quad \Omega\times(0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u = u_0 & \text{on} \quad \Omega\times\{0\}. \end{cases}$$

Here u and p are unknown functions which represent the velocity and the pressure of fluid respectively. f and u_0 are given functions which stand for the external force and the initial velocity respectively. It is convenient for analysis to rewrite the above system of partial differential equations to an ordinary equation on Banach space as usual manner. Function spaces $\mathcal{V}$, H, V and V' are defined by

$$\begin{cases} \mathcal{V} = \{\varphi \in C_0^\infty(\Omega) \mid \operatorname{div} \varphi = 0\}, \\ H = \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega), \\ V = \text{the closure of } \mathcal{V} \text{ in } H_0^1(\Omega), \\ V' = \text{the dual space of } V \text{ with respect to } L^2(\Omega)\text{-inner product.} \end{cases}$$

Then our problem can be written in the abstract form as

$$\text{(2.1)} \qquad \begin{cases} \dfrac{du}{dt} + Au + Bu = f \quad \text{in } \ V' \ \text{ for almost every } \ t \in (0,T), \\ u(\cdot, 0) = u_0 \in H. \end{cases}$$

Here A and B are respectively linear and non-linear operators from V to V' defined by

$$\begin{cases} {}_{V'}\langle Au, \varphi \rangle_V = \displaystyle\int_\Omega \langle\!\langle \nabla u, \nabla \varphi \rangle\!\rangle dx, \\ {}_{V'}\langle Bu, \varphi \rangle_V = \displaystyle\int_\Omega \langle (u \cdot \nabla) u, \varphi \rangle dx \end{cases}$$

for $\varphi \in V$. Notations ${}_{V'}\langle \cdot, \cdot \rangle_V$, $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\langle \cdot, \cdot \rangle$ mean respectively the duality between elements of V' and V, the pointwise inner products between $m \times m$-tensors and between m-dimensional vectors. Now we define the concept of a *weak solutions*

Definition. We suppose $u_0 \in H$ and $f \in L^2_{\mathrm{loc}}([0,\infty); V')$. We say that u is a *weak solution* to (2.1) if it belongs to $L^2(0,T;V)$ with the time derivative $\dfrac{du}{dt} \in L^1(0,T;V')$ and satisfies (2.1).

In the sequel we shall give an alternative approach constructing weak solutions, which is successful if Ω as a two- or three-dimensional bounded domain.

3 Discretization

In this section we explain our scheme. We employ the partially implicit scheme of discretization of (2.1) with respect to the time variable

$$
\begin{cases} \dfrac{u_n^h - u_{n-1}^h}{h} + Au_n^h + Bu_{n-1}^h = f_n^h \quad \text{in } V', \\ \\ u_0^h = u_0, \end{cases} \tag{3.1}
$$

where

$$f_n^h = \frac{1}{h}\int_{(n-1)h}^{nh} f(t)dt.$$

The above equation can be considered as the Euler-Lagrange equation to the functional

$$I^h(u) = \frac{\|u - u_{n-1}^h\|_2^2}{h} + \|\nabla u\|_2^2 + 2b(u_{n-1}^h, u_{n-1}^h, u) - 2_{V'}\langle f_n^h, u\rangle_V \quad (h > 0)$$

on V, where

$$b(u, v, w) = \int_\Omega \langle (u \cdot \nabla)v, w\rangle dx \quad \text{for } u,\ v,\ w \in V.$$

For fixed $h > 0$, we can obtain the minimizer u_n^h of $I(u)$ on V, that is the discrete Morse semiflow. However I cannot show a priori estimates uniformly on h. Therefore I cannot show the convergence as $h \downarrow 0$. Hence we need some modification to the functional. We modify $I^h(u)$ to

$$J^h(u) = \frac{\|u - u_{n-1}^h\|_2^2}{h} + \|\nabla u\|_2^2 + 2\rho(b(u_{n-1}^h, u_{n-1}^h, u)) - 2_{V'}\langle f_n^h, u\rangle_V \quad (h > 0),$$

where ρ is a truncating function satisfying

$$\rho(x) = \begin{cases} x & \text{for} \quad x \in [-1, \infty), \\ \\ 0 & \text{for} \quad x \in (-\infty, -2]. \end{cases}$$

Let $\{u_n^h\}$ be the minimizer of J^h on V, which is obtained by the standard minimizing sequence argument:

$$u_n^h \ :\ J^h(u) \to \min \text{ on } V. \tag{3.2}$$

The Euler-Lagrange equation to this functional is

$$\frac{u_n^h - u_{n-1}^h}{h} + Au_n^h + \rho'(b(u_{n-1}^h, u_{n-1}^h, u_n^h))Bu_{n-1}^h = f_n^h \quad \text{in } V'. \tag{3.3}$$

If u_n^h converges to some functional u as $h \downarrow 0$, we can expect for small h

$$b(u_{n-1}^h, u_{n-1}^h, u_n^h) \approx b(u_{n-1}^h, u_{n-1}^h, u_{n-1}^h) = 0.$$

And ρ is an identity function near $x = 0$. Therefore (3.3) may be a good approximation of (3.1). Indeed this scheme gives weak solutions as $h \downarrow 0$ in (3.3). We give the detail in the next section.

4 Results

Let assume $u_0 \in V$. The using the standard argument of minimizing sequences, we find the sequence $\{u_n^h\}$ can be defined. First we give its a priori estimates.

Lemma 4.1. *It holds that*

$$||u_n^h||_2^2 + \sum_{k=1}^n ||u_k^h - u_{k-1}^h||_2^2 + \sum_{k=1}^n h||\nabla u_k^h||_2^2 \leq ||u_0||_2^2 + C_1 nh + C_2 \int_0^{nh} ||f||_{V'}^2 dt.$$

Proof. We take $u_n^h \in V$ as a test function for the Euler-Lagrange equation (3.3). It follows from the choice of ρ that

$$-2\rho'(b(u_{n-1}^h, u_{n-1}^h, u_n^h))b(u_{n-1}^h, u_{n-1}^h u_n^h) \leq -2 \min \rho'(x)x = C_1 < \infty.$$

Combining this with the Poincaré inequality we have

$$\begin{aligned}
&||u_n^h||_2^2 + \sum_{k=1}^n ||u_k^h - u_{k-1}^h||_2^2 + 2\sum_{k=1}^n h||\nabla u_k^h||_2^2 \\
&\leq ||u_0||_2^2 + C_1 nh + 2\sum_{k=1}^n h||f_k^h||_{V'}||u_k^h||_V \\
&\leq ||u_0||_2^2 + C_1 nh + \sum_{k=1}^n h||\nabla u_k^h||_2^2 + C_2 \sum_{k=1}^n h||f_k^h||_{V'}^2.
\end{aligned}$$

Taking $\sum_{k=1}^n h||f_k^h||_{V'}^2 \leq \int_0^{nh} ||f||_{V'}^2 dt$ into consideration, we obtain the assertion. □

Next we give an estimate for the finite difference in time variable of the approximate solution. From now we frequently use the Gagliardo-Nirenberg inequality

$$||u||_4 \leq C_{GN} ||u||_2^\theta ||\nabla u||_2^{1-\theta} \quad \text{for} \quad u \in H_0^1(\Omega),$$

where $\theta = \dfrac{1}{2}$ when $m = 2$, and $\theta = \dfrac{1}{4}$ when $m = 3$. Here $\|\cdot\|_p$ is the $L^p(\Omega)$-norm.

Lemma 4.2. *Let $\gamma = \dfrac{1}{1-\theta}$, i.e., $\gamma = 2$ when $m = 2$, and $\gamma = \dfrac{4}{3}$ when $m = 3$. Then it holds that*

$$\sum_{k=1}^{n} h \left\| \frac{u_k^h - u_{k-1}^h}{h} \right\|_{V'}^{\gamma} \le C_3 \left(1 + nh + \int_0^{nh} \|f\|_{V'}^2 dt \right)^{\gamma}.$$

Proof. It follows from (3.3) that

$$\left| \int_\Omega \frac{\langle u_k^h - u_{k-1}^h, \varphi \rangle}{h} dx \right|$$

$$\le \|\nabla u_k^h\|_2 \|\nabla \varphi\|_2 + C_4 C_{GN}^2 \|\rho'\|_\infty \|u_{k-1}^h\|_2^{2\theta} \|\nabla u_{k-1}^h\|_2^{2(1-\theta)} \|\nabla \varphi\|_2 + \|f_k^h\|_{V'} \|\varphi\|_V$$

for any $\varphi \in V$. Therefore by Lemma 4.1 we have

$$\sum_{k=1}^{n} h \left\| \frac{u_k^h - u_{k-1}^h}{h} \right\|_{V'}^{\gamma}$$

$$\le C_5 \left\{ \sum_{k=1}^{n} h \left(\|\nabla u_k^h\|_2^2 + 1 \right) + \sup_{0 \le \ell \le n-1} \|u_\ell^h\|_2^{2\theta\gamma} \sum_{k=0}^{n-1} h \|\nabla u_k^h\|_2^2 + \sum_{k=1}^{n} h \left(\|f_k^h\|_{V'}^2 + 1 \right) \right\}$$

$$\le C_3 \left(1 + nh + \int_0^{nh} \|f\|_{V'}^2 dt \right)^{\gamma}.$$

□

Let u^h, $\bar{u}^h$ and $\tilde{u}^h$ be

$$\begin{cases} u^h(x,t) = \dfrac{t-(n-1)h}{h} u_n^h(x) + \dfrac{nh-t}{h} u_{n-1}^h(x), \\[2mm] \bar{u}^h(x,t) = u_n^h(x), \\[2mm] \tilde{u}^h(x,t) = u_{n-1}^h(x) \end{cases}$$

for $t \in ((n-1)h, nh]$. Then it follows from Lemmata 4.1 and 4.2 that

$$\begin{cases} \{u^h\}, \{\tilde{u}^h\}, \{\bar{u}^h\} \text{ are bounded sets in } L^\infty_{\mathrm{loc}}([0,\infty); H) \cap L^2_{\mathrm{loc}}([0,\infty); V), \\[2mm] \left\{ \dfrac{du^h}{dt} \right\} \quad \text{is a bounded set in } L^\gamma_{\mathrm{loc}}([0,\infty); V'). \end{cases}$$

Hence we can extract a subsequence of h so that the functions converge.

Proposition 4.1. *The functions u^h, $\bar{u}^h$ and $\tilde{u}^h$ converge to a function u in the sense that*

$$\begin{aligned}
u^h \to u \quad & \text{weakly star in } L^\infty_{\mathrm{loc}}([0,\infty); H), \text{ weakly in } L^2_{\mathrm{loc}}([0,\infty); V), \\
& \text{and strongly in } L^2_{\mathrm{loc}}([0,\infty); H) \cap L^2_{\mathrm{loc}}([0,\infty); L^4(\Omega)), \\
\bar{u}^h \to u \quad & \text{weakly star in } L^\infty_{\mathrm{loc}}([0,\infty); H), \text{ weakly in } L^2_{\mathrm{loc}}([0,\infty); V), \\
& \text{and strongly in } L^2_{\mathrm{loc}}([0,\infty); H) \cap L^2_{\mathrm{loc}}([0,\infty); L^4(\Omega)), \\
\tilde{u}^h \to u \quad & \text{weakly star in } L^\infty_{\mathrm{loc}}([0,\infty); H), \text{ weakly in } L^2_{\mathrm{loc}}([0,\infty); V), \\
& \text{and strongly in } L^2_{\mathrm{loc}}([0,\infty); H) \cap L^2_{\mathrm{loc}}([0,\infty); L^4(\Omega)) \\
\frac{du^h}{dt} \to \frac{du}{dt} \quad & \text{weakly in } L^\gamma_{\mathrm{loc}}([0,\infty); V').
\end{aligned}$$

as $h \downarrow 0$ up to a subsequence.

Proof. First we show

$$u^h - \bar{u}^h \to 0 \quad \text{and} \quad u^h - \tilde{u}^h \to 0 \tag{4.1}$$

as $h \downarrow 0$ in $L^2_{\mathrm{loc}}([0,\infty); H)$. Since

$$\begin{cases} u^h - \bar{u}^h = \dfrac{t-kh}{h}(u^h_k - u^h_{k-1}), \\[2ex] u^h - \tilde{u}^h = \dfrac{t-(k-1)h}{h}(u^h_k - u^h_{k-1}) \end{cases}$$

for $t \in ((k-1)h, kh]$, it holds that

$$\begin{aligned}
\int_0^T \|u^h - \hat{u}^h\|_2^2 dt &\le \sum_{k=1}^{\lceil T/h \rceil} h\|u^h_k - u^h_{k-1}\|_2^2 \\
&\le h\left\{\|u_0\|_2^2 + C_1(T+h) + C_2\int_0^{T+h} \|f\|_{V'}^2 dt\right\} \to 0 \quad \text{as} \quad h \downarrow 0,
\end{aligned}$$

where $\hat{u}^h$ is $\bar{u}^h$ or $\tilde{u}^h$, and $\lceil T/h \rceil$ is the ceiling of T/h, *i.e.*, the smallest integer greater than or equal to T/h. Here we use Lemma 4.1.

Therefore the result is derived from the standard weak (star) compactness result of Banach spaces, [9, Chapter III, Theorem 2.1], the diagonal argument, and (4.1). □

In consequence of Proposition 4.1 the convergence

$$\frac{du^h}{dt} \rightharpoonup u_t \quad \text{in } L^\gamma_{\mathrm{loc}}([0,\infty);V'),$$

$$A\bar{u}^h \rightharpoonup Au \quad \text{in } L^2_{\mathrm{loc}}([0,\infty);V'),$$

$$B\tilde{u}^h \rightharpoonup Bu \quad \text{in } L^2_{\mathrm{loc}}([0,\infty);V')$$

hold along the subsequence. However to show the convergence

$$\rho'(b(u^h_{n-1}, u^h_{n-1}, u^h_n) \to 1$$

we need the compactness of imbedding $H^1_0(\Omega) \hookrightarrow L^4(\Omega)$ and so on. Therefore we must assume Ω is bounded two- or three-dimensional domain.

Proposition 4.2. *It holds that*

$$\rho'(b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h))B\tilde{u}^h \to Bu$$

weakly in $L^\gamma_{\mathrm{loc}}([0,\infty);V')$ *as* $h \downarrow 0$ *up to a subsequence.*

Proof. Let $\gamma' = \dfrac{\gamma}{\gamma-1}$, *i.e.*, $\gamma' = 2$ when $m = 2$, and $\gamma' = 4$ when $m = 3$. For the purpose we put

$$\rho'(b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h))B\tilde{u}^h - Bu = \mathrm{I}^h + \mathrm{II}^h,$$

where

$$\begin{cases} \mathrm{I}^h = \rho'(b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h))(B\tilde{u}^h - Bu), \\ \mathrm{II}^h = (\rho'(b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h)) - 1)Bu. \end{cases}$$

Let $\Phi \in C^\infty(\bar{\Omega} \times [0,T])$ be a function satisfying

$$\Phi(\cdot,t) \in \mathcal{V} \quad \text{for } t \in [0,T];$$

the set of such functions is dense in $L^{\gamma'}(0,T;V)$. By use of [9, Chapter III, Lemma 3.2] and Proposition 4.1 we have

$$\left| \int_0^T {}_{V'}\langle \mathrm{I}^h, \Phi\rangle_V dt \right| \le C_6 \|\rho'\|_\infty \int_0^T \|\tilde{u}^h - u\|_2 \left(\|\tilde{u}^h\|_2 + \|u\|_2\right) \|\nabla\Phi\|_\infty dt \to 0 \quad \text{as} \quad h \downarrow 0,$$

which shows the weak convergence of I^h to 0 in $L^2(0,T;V')$.

Next we show the weak convergence of II^h. By the facts $\rho' \in L^\infty(\mathbf{R})$ and $1 \le \|\rho'\|_\infty$ we have

$$\left|{}_{V'}\langle \mathrm{II}^h, \Phi\rangle_V\right| \le 2\|\rho'\|_\infty |b(u,u,\Phi)| \le C_7 C^2_{GN} \|\rho'\|_\infty \|u\|_2^{2\theta} \|\nabla u\|_2^{2(1-\theta)} \|\nabla\Phi\|_2 \in L^1(0,T)$$

for $\Phi \in L^{\gamma'}(0,T;V)$. On the other hand by use of [9, Chapter II, Lemma 1.3] we have

$$\left|{}_{V'}\langle \mathrm{II}^h, \Phi\rangle_V\right| = |(\rho'(b(\tilde{u}^h,\tilde{u}^h,\bar{u}^h)) - \rho'(b(\tilde{u}^h,\tilde{u}^h,\tilde{u}^h)))b(u,u,\Phi)|$$

$$\leq C_8\|\rho''\|_\infty\|\tilde{u}^h\|_4\|\nabla\tilde{u}^h\|_2\|\bar{u}^h - \tilde{u}^h\|_4|b(u,u,\Phi)|.$$

With the help of Proposition 4.1 by extracting a subsequence again, if necessary, $\|u^h(t)\|_4$ converges to $\|u(t)\|_4$ for almost every $t \in (0,T)$, and especially $\sup_h \|u^h(t)\|_4$ is finite (of course the supremum may depend on t). Moreover it holds that for every $\varepsilon > 0$

$$\int_0^T \|\nabla\tilde{u}^h\|_2\|\bar{u}^h - \tilde{u}^h\|_4 dt \leq \varepsilon\int_0^T \|\nabla\tilde{u}^h\|_2^2 dt + \frac{1}{4\varepsilon}\int_0^T \|\bar{u}^h - \tilde{u}^h\|_4^2 dt \to C_9(T)\varepsilon$$

as $h \downarrow 0$, which implies $\|\nabla\tilde{u}^h\|_2\|\bar{u}^h - \tilde{u}^h\|_4 \to 0$ as $h \downarrow 0$ for almost every $t \in (0,T)$ provided, if necessary, we extract a subsequence again. These facts yield

$$\left|{}_{V'}\langle \mathrm{II}^h, \Phi\rangle_V\right| \to 0 \quad \text{as} \quad h \downarrow 0 \quad \text{for almost every} \quad t \in (0,T). \tag{4.2}$$

Hence the dominated convergence theorem implies

$$\int_0^T {}_{V'}\langle \mathrm{II}^h, \Phi\rangle_V dt \to 0 \quad \text{as} \quad h \downarrow 0.$$

□

Finally we must show that the initial condition is satisfied.

Proposition 4.3. *It holds that*

$$u(0) = u_0.$$

Proof. It holds that

$$\begin{aligned}
&\|u(0) - u_0\|_{V'} \\
&\leq \|u(0) - u(t_j)\|_{V'} + \|u(t_j) - u^h(t_j)\|_{V'} + \|u^h(t_j) - u_0\|_{V'} \\
&\leq t_j^{1/\gamma'}\left(\int_0^{t_j}\left\|\frac{du}{dt}\right\|_{V'}^\gamma dt\right)^{1/\gamma} + \|u(t_j) - u^h(t_j)\|_{V'} + t_j^{1/\gamma'}\left(\int_0^{t_j}\left\|\frac{d^\varepsilon u}{dt}\right\|_{V'}^\gamma dt\right)^{1/\gamma} \\
&= O\left(t_j^{1/\gamma'}\right) \quad \text{as} \quad \varepsilon \downarrow 0 \\
&\to 0 \quad \text{as} \quad j \to \infty,
\end{aligned}$$

whence $u(0) = u_0$ is derived. □

Consequently we conclude that

Theorem 4.1. *Our scheme* (3.2) *gives the Leray-Hopf weak solution as $h \downarrow 0$ along a subsequence, if $\Omega \subset \mathbf{R}^m$ ($m = 2$ or 3) is bounded and $u_0 \in V$, $f \in L^2_{\text{loc}}([0,\infty); V')$.*

By use of (3.3) and the argument similar to the proof of Proposition 4.2 we have the energy equality for the two-dimensional flow and the energy inequality for the three-dimensional flow.

Theorem 4.2. *When $m = 2$, our weak solution satisfies the energy equality*

$$\|u(\cdot,t)\|_2^2 + 2\int_0^t \|\nabla u(\cdot,\tau)\|_2^2 d\tau = \|u_0\|_2^2 + 2\int_0^t {}_{V'}\langle f(\cdot,\tau), u(\cdot,\tau)\rangle_V d\tau \tag{4.3}$$

for any $t \in [0,\infty)$. When $m = 3$, it satisfies the energy inequality

$$\|u(\cdot,t)\|_2^2 + 2\int_0^t \|\nabla u(\cdot,\tau)\|_2^2 d\tau \le \|u_0\|_2^2 + 2\int_0^t {}_{V'}\langle f(\cdot,\tau), u(\cdot,\tau)\rangle_V d\tau \tag{4.4}$$

for almost every $t \in [0,\infty)$.

Proof. When $m = 2$, because of $u \in L^2_{\text{loc}}([0,\infty); V)$ and $u_t \in L^2_{\text{loc}}([0,\infty); V')$, we have (4.3) by [9, Chapter III, Lemma 1.2].

Finally we shall show (4.4) for $m = 3$. We take $2hu_n^h \in V$ as a test function in (3.3), and sum up with respect to n. We use estimates

$$0 \le \sum_{k=1}^{n} \|u_k^h - u_{k-1}^h\|_2^2$$

and

$$-2\rho'(b(u_{k-1}^h, u_{k-1}^n, u_k^h))b(u_{k-1}^h, u_{k-1}^n, u_k^h) \le 2\left\{\rho'(b(u_{k-1}^h, u_{k-1}^n, u_k^h))b(u_{k-1}^h, u_{k-1}^n, u_k^h)\right\}_-,$$

where $g_- = \max\{-g, 0\}$. Then we get

$$\begin{aligned}&\|u^h(\cdot,nh)\|_2^2 + 2\int_0^{nh} \|\nabla \bar{u}^h(\cdot,\tau)\|_2^2 d\tau \\ &\le \|u_0\|_2^2 + 2\int_0^{nh} \left\{\rho'(b(\tilde{u}^h,\tilde{u}^h,\bar{u}^h))b(\tilde{u}^h,\tilde{u}^h,\bar{u}^h)\right\}_- d\tau + 2\int_0^{nh} {}_{V'}\langle f(\cdot,\tau), \bar{u}^h(\cdot,\tau)\rangle_V d\tau\end{aligned}$$

in terms of u^h, $\bar{u}^h$ and $\tilde{u}^h$. Let $t \in (0,\infty)$ be fixed, and n be an integer such that

$$\left[\frac{t}{h}\right] \le n \le \left\lceil\frac{t}{h}\right\rceil.$$

The estimate

$$0 \le 2\left\{\rho'(b(u^h_{k-1}, u^n_{k-1}, u^h_k))b(u^h_{k-1}, u^n_{k-1}, u^h_k)\right\}_- \le C_1$$

holds. Moreover since $b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h) = b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h - \tilde{u}^h)$, we have

$$2\left\{\rho'(b(u^h_{k-1}, u^n_{k-1}, u^h_k))b(u^h_{k-1}, u^n_{k-1}, u^h_k)\right\}_-$$

$$\le \begin{cases} C_{10}\|\rho'\|_\infty \|\tilde{u}^h\|_4 \|\nabla\tilde{u}^h\|_2 \|\bar{u}^h - \tilde{u}^h\|_4 & \text{if } b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h) \in \text{supp } [\rho'(x)x]_-, \\ 0 & \text{otherwise}, \end{cases}$$

$$\to 0 \quad \text{as } h \downarrow 0 \quad \text{for almost every } t \in (0,T)$$

by Proposition 4.1. Here we need an argument similar to that to derive (4.2). Therefore the bounded convergence theorem yields

$$2\int_0^{nh} \left\{\rho'(b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h))b(\tilde{u}^h, \tilde{u}^h, \bar{u}^h)\right\}_- d\tau \to 0 \quad \text{as } h \downarrow 0.$$

By this fact and the argument in [9, Chapter III, Remark 4.1] we have the energy inequality by passing to the limit $h \downarrow 0$. □

By approximate argument of the initial value it folds the same result as Theorems 4.1 and 4.2 even for $u_0 \in H$

Theorem 4.3. *Our scheme* (3.2) *still works even for* $u_0 \in H$ *with further suitable modification.*

For the details see [4].

5 Final remarks

Of course the existence of weak solution has been already well known. But I think our scheme (3.2) has potential interest. For minimizing property may clarify the structure of partial regularity of weak solutions by virtue of technique of Giaquinta-Giusti [2, 3].

Our scheme also works for the problem with non-homogeneous boundary condition, if $\partial\Omega$ is suitably smooth.

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